

PRETTY CLEAN MONOMIAL IDEALS AND LINEAR QUOTIENTS

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ABSTRACT. We study basic properties of monomial ideals with linear quotients. It is shown that if the monomial ideal I has linear quotients, then the squarefree part of I and each component of I as well as \mathfrak{m}/I have linear quotients, where \mathfrak{m} is the graded maximal ideal of the polynomial ring. As an analogy to the Rearrangement Lemma of Björner and Wachs we also show that for a monomial ideal with linear quotients the admissible order of the generators can be chosen degree increasingly.

As a generalization of the facet ideal of a forest, we define monomial ideals of forest type and show that they are pretty clean. This result recovers a recent result of Tuly and Villarreal about the shellability of a clutter with the free vertex property. As another consequence of this result we show that if I is a monomial ideal of forest type, then Stanley's conjecture on Stanley decomposition holds for S/I . We also show that a clutter is totally balanced if and only if it has the free vertex property.

INTRODUCTION

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables, and $I \subset S$ a monomial ideal. We denote by $G(I)$ the unique minimal monomial system of generators of I . We say that I has linear quotients, if there exists an order $\sigma = u_1, \dots, u_m$ of $G(I)$ such that the ideal $(u_1, \dots, u_{i-1}) : u_i$ is generated by a subset of the variables for $i = 2, \dots, m$. We denote this subset by $q_{u_i, \sigma}(I)$. Any order of the generators for which we have linear quotients will be called an admissible order. Ideals with linear quotients were introduced by Herzog and Takayama [15]. If each component of I has linear quotients, then we say I has componentwise linear quotients.

The concept of linear quotients, similarly as the concept of shellability, is purely combinatorial. However both concepts have strong algebraic implications. Indeed, an ideal with linear quotients has componentwise linear resolutions while shellability of a simplicial complex implies that it is sequentially Cohen-Macaulay. These similarities are not accidental. In fact, let Δ be a simplicial complex and I_Δ its Stanley-Reisner ideal. It is well-known that I_Δ has linear quotients if and only if the Alexander dual of Δ is shellable. Thus at least in the squarefree case “linear quotients” and “shellability” are dual concepts. On the other hand, linear quotients are not only defined for squarefree monomial ideals, and hence this concept is more general than that of shellability.

In this paper we prove some fundamental properties of monomial ideals with linear quotients. In general, the product of two ideals with linear quotients need not to have

1991 *Mathematics Subject Classification.* 13F20, 13F55, 13A30, 16W70.

Key words and phrases. linear quotients, pretty clean modules, shellability, Stanley decomposition.

The second author is grateful for the financial support by DFG (Deutsche Forschungsgemeinschaft) during the preparation of this work.

linear quotients, even if one of them is generated by a subset of the variables, see Example 2.4. However in Lemma 2.5, we show that if $I \subset S$ is a monomial ideal with linear quotients, then $\mathfrak{m}I$ has linear quotients, where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of S .

Let I be a monomial ideal with linear quotients and $\sigma = u_1, \dots, u_m$ an admissible order of $G(I)$. It is not hard to see that $\deg u_i \geq \min\{\deg u_1, \dots, \deg u_{i-1}\}$, for all $i \in [m] = \{1, \dots, m\}$. But this order need not to be a degree increasing order. We show in Lemma 2.1, that there exists a degree increasing admissible order σ' induced by σ . Furthermore, one has $q_{u,\sigma}(I) = q_{u,\sigma'}(I)$ for any $u \in G(I)$, see Proposition 2.2. This implies in particular the “Rearrangement Lemma” of Björner and Wachs [2].

As a main result of Section 2, we show in Theorem 2.7, that any monomial ideal with linear quotients has componentwise linear quotients, and hence it is componentwise linear. Conversely, assuming that all components of I have linear quotients, we can prove that I has linear quotients only under some extra assumption, see Proposition 2.9. It would be of interest to know whether the converse of Theorem 2.7 is true in general.

Herzog and Hibi showed in [8] that a squarefree monomial ideal I is componentwise linear if and only if the squarefree part of each component has a linear resolution. We would like to remark that the “only if” part of this statement is true more generally. Indeed for *any* componentwise linear monomial ideal, the squarefree part of each component has a linear resolution. Here we prove a slightly different result by showing that if a monomial ideal I has linear quotients, then the squarefree part of I has linear quotients. This together with Theorem 2.7 implies that the squarefree part of each component of I has again linear quotients. As a corollary of the above facts we obtain that if Δ is shellable, then each facet skeleton (see the definition in Section 2) of Δ is shellable. Unless Δ is pure, this result differs from the well-known fact that each skeleton of a shellable simplicial complex is again shellable.

In Section 3, we give a large and combinatorially interesting class \mathcal{J} of monomial ideals which are pretty clean (Theorem 3.4), and hence Stanley’s conjecture on Stanley decompositions [20] holds for S/I . As another consequence of Theorem 3.4 we get the main result of [7], which says that $S/I(\Delta)$ is sequentially Cohen-Macaulay for any forest Δ , as defined by Faridi [5]. The class \mathcal{J} is a non squarefree version of the class of facet ideals of forests. Any ideal in \mathcal{J} is called a monomial ideal of forest type. We show in Theorem 3.8 that I is a monomial ideal of forest type if and only if I has the free variable property. Identifying a squarefree monomial ideal with a clutter, Theorem 3.8 says that a clutter has the free vertex property in the sense of Tuyl and Villarreal if and only if the clutter corresponds to a forest in the sense of Faridi, equivalently, a totally balanced clutter in the language of hypergraphs. Let \mathcal{C} be a clutter, and let $\Delta_{\mathcal{C}}$ be the simplicial complex whose Stanley–Reisner ideal is the edge ideal of \mathcal{C} . In [23, Theorem 5.3] Villarreal and Tuyl show that $\Delta_{\mathcal{C}}$ is shellable if \mathcal{C} has the free vertex property. Therefore Theorem 3.4 may be viewed as a generalization of [23, Theorem 5.3].

In the last section we give some examples of quasi-forests. These examples show that the facet ideal of a quasi-forest need not always to be clean. It would be interesting to classify all quasi-forests whose facet ideals are clean.

1. PRELIMINARIES AND BACKGROUND

In this section we fix the terminology, review some notation on simplicial complexes and setup some background.

A *simplicial complex* Δ over a set of vertices $[n] = \{1, \dots, n\}$ is a collection of subsets of $[n]$ with the property that $i \in \Delta$ for all $i \in [n]$, and if $F \in \Delta$ then all the subsets of F are also in Δ (including the empty set). An element of Δ is called a *face* of Δ , and the maximal faces of Δ under inclusion are called *facets*. We denote $\mathcal{F}(\Delta)$ the set of facets of Δ . The simplicial complex with facets F_1, \dots, F_m is denoted by $\langle F_1, \dots, F_m \rangle$. The *dimension* of a face F is defined as $|F| - 1$, where $|F|$ is the number of vertices of F . The dimension of the simplicial complex Δ is the maximal dimension of its facets. A simplicial complex Γ is called a *subcomplex* of Δ if $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$.

A subset C of $[n]$ is called a *vertex cover* of Δ , if $C \cap F \neq \emptyset$ for all facets F of Δ . A vertex cover C is said to be *minimal* if no proper subset of C is a vertex cover of Δ . Recently, vertex cover algebra was studied in [9] and [10].

We denote by $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables over a field K . To a given simplicial complex Δ on the vertex set $[n]$, the Stanley–Reisner ideal, whose generators correspond to the non-faces of Δ is well studied, see for example in [20], [1] and [12] for details. Another squarefree monomial ideal associated to Δ , so-called facet ideal, was first studied by Faridi [5]. The ideal $I(\Delta)$ generated by all monomials $x_{i_1} \cdots x_{i_s}$ where $\{i_1, \dots, i_s\}$ is a facet of Δ , is called the *facet ideal* of Δ . For a simplicial complex of dimension 1, the facet ideal is the *edge ideal*, which was first studied by Villarreal [24].

The following definitions were first introduced by Faridi in [5]. Let Δ be a simplicial complex. A facet F of Δ is called a *leaf* if either F is the only facet of Δ , or there exists a facet $G \neq F$ in Δ such that $F \cap H \subseteq F \cap G$ for any facet $H \in \Delta$, $H \neq F$. The facet G is called a *branch* of F . A simplicial complex Δ is called a *tree* if it is connected and every nonempty subcomplex of Δ has a leaf. A simplicial complex Δ with the property that every connected component is a tree is called a *forest*. A vertex $t \in F$ is called a *free vertex* of F if $F \in \mathcal{F}(\Delta)$ is the unique facet which contains t . It is easy to see that any leaf has a free vertex.

Recall that the *Alexander dual* Δ^\vee of a simplicial complex Δ is the simplicial complex whose faces are $\{[n] \setminus F : F \notin \Delta\}$. Let I be a squarefree monomial ideal in S . We denote by I^\vee the squarefree monomial ideal which minimally generated by all monomials $x_{i_1} \cdots x_{i_k}$, where $(x_{i_1}, \dots, x_{i_k})$ is a minimal prime ideal of I . It is easy to see that for any simplicial complex Δ , one has $I_{\Delta^\vee} = (I_\Delta)^\vee$. Let $\Delta^c = \langle [n] \setminus F : F \in \mathcal{F}(\Delta) \rangle$. Then $I_{\Delta^\vee} = I(\Delta^c)$, see [11].

For any set $U \subset [n]$, we denote $u = \prod_{j \in U} x_j$ the squarefree monomial in S whose support is U . In general, for any monomial $u \in S$, the *support* of u is $\text{supp}(u) = \{j : x_j \mid u\}$.

Remark 1.1. Let Δ be a simplicial complex on $[n]$. Then

$$G(I(\Delta)^\vee) = \{u = \prod_{j \in U} x_j : \text{where } U \text{ is a minimal vertex cover of } \Delta\}.$$

Now we recall the definition of clean and pretty clean modules of the type S/I , where $I \subset S$ is a monomial ideal. According to [13], a filtration $\mathcal{F} : I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ of S/I is called a *pretty clean filtration* if

- (a) for all j one has $I_j/I_{j-1} \cong S/P_j$ where P_j is a monomial prime ideal;
- (b) for all $i < j$, if $P_i \subset P_j$, then $P_i = P_j$.

The set of prime ideals $\{P_1, \dots, P_r\}$ is called the *support* of \mathcal{F} and denoted by $\text{Supp}(\mathcal{F})$. The module S/I is called *pretty clean* if it has a pretty clean filtration.

Dress [4] calls the ring S/I *clean*, if there exists a chain of ideals as above such that all the P_i are minimal prime ideals of I . By an abuse of notation we call I (pretty) clean if S/I is (pretty) clean. Obviously, any clean ideal is pretty clean. If I is a squarefree monomial ideal, then pretty clean implies also clean. The following fact was first shown by Dress.

Theorem 1.2. [4] *Let Δ be a simplicial complex and $I = I_\Delta \subset S$ its Stanley-Reisner ideal. Then the simplicial complex Δ is (non-pure) shellable if and only if I_Δ is clean.*

The following notion is important for our later discussion. Let $I = (u_1, \dots, u_m)$ be a monomial ideal in S . According to [15], the monomial ideal I has linear quotients if one can order the set of minimal generators of I , $G(I) = \{u_1, \dots, u_m\}$, such that the ideal $(u_1, \dots, u_{i-1}) : u_i$ is generated by a subset of the variables for $i = 2, \dots, m$. This means for each $j < i$, there exists a $k < i$ such that $u_k : u_i = x_t$ and $x_t \mid u_j : u_i$, where $t \in [n]$ and $u_k : u_i = u_k / \gcd(u_k, u_i)$. In the case that I is squarefree, it is enough to show that for each $j < i$, there exists a $k < i$ such that $u_k : u_i = x_t$ and $x_t \mid u_j$. Such an order of generators is called an *admissible order* of $G(I)$. Let $\sigma = u_1, \dots, u_m$ be an admissible order of $G(I)$. We denote by $q_{u_j, \sigma}(I) \subset \{x_1, \dots, x_n\}$ the set of minimal generators of $(u_1, \dots, u_{j-1}) : u_j$.

It is known that if I is a monomial ideal with linear quotients and generated in one degree, then I has a linear resolution. See for example in [25] an easy proof.

Remark 1.3. For an ideal which has linear quotients, there might exist several admissible orders. For example, let $I = (x_1x_2, x_1x_3^2x_4, x_2x_4) \subset K[x_1, x_2, x_3, x_4]$. Then $\sigma_1 = x_1x_2, x_1x_3^2x_4, x_2x_4$ and $\sigma_2 = x_1x_2, x_2x_4, x_1x_3^2x_4$ both are admissible orders of $G(I)$.

The following result relates squarefree monomial ideals with linear quotients to (non-pure) shellable simplicial complexes. The concept non-pure shellability was first defined by Björner and Wachs [2, Definition 2.1].

Theorem 1.4. [11, Theorem 1.4] *Let Δ be a simplicial complex and $(I_\Delta)^\vee$ the Alexander dual of its Stanley-Reisner ideal. Then Δ is (non-pure) shellable if and only if $(I_\Delta)^\vee$ has linear quotients.*

Combining Theorem 1.2 and Theorem 1.4, we get the following

Corollary 1.5. *Let $I \subset S$ be a squarefree monomial ideal. Then I is clean if and only if I^\vee has linear quotients.*

2. MONOMIAL IDEALS WITH LINEAR QUOTIENTS

In this section we prove some fundamental properties of ideals with linear quotients.

Let $I \subset S$ be a monomial ideal with linear quotients and u_1, \dots, u_m an admissible order of $G(I)$. It is easy to see that $\deg u_i \geq \min\{\deg u_1, \dots, \deg u_{i-1}\}$ for $i = 2, \dots, m$. In particular, $\deg u_1 = \min\{\deg u_1, \dots, \deg u_m\}$. But in general, this order need not to be a degree increasing order. For example, the ideal $I = (x_1x_2, x_1x_3^2x_4, x_2x_4)$ has linear quotients in the given order, but $\deg x_1x_3^2x_4 > \deg x_2x_4$.

In the following lemma we show that for any ideal with linear quotients there exists an admissible order u_1, \dots, u_m of $G(I)$ such that $\deg u_i \leq \deg u_{i+1}$ for $i = 1, \dots, m-1$. We call such an order a *degree increasing admissible order*.

Lemma 2.1. *Let $I \subset S$ be a monomial ideal with linear quotients. Then there is a degree increasing admissible order of $G(I)$.*

Proof. We use induction on m , the number of generators of I , to prove the statement. If $m = 1$, there is nothing to show.

Assume $m > 1$ and u_1, \dots, u_m is an admissible order. It is clear that $J = (u_1, \dots, u_{m-1})$ has linear quotients with the given order. By induction hypothesis, we may assume that $\deg u_i \leq \deg u_{i+1}$ for $i = 1, \dots, m-2$. Assume that $\deg u_{m-1} > \deg u_m$. Let $j+1$ be the smallest integer such that $\deg u_{j+1} > \deg u_m$. By the observation before this lemma, one sees that $j+1 \neq 1$. Now we show that $u_1, \dots, u_j, u_m, u_{j+1}, \dots, u_{m-1}$ is an admissible order which is obviously degree increasing.

We need to prove that $(u_1, \dots, u_j) : u_m$ and $(u_1, \dots, u_j, u_m, u_{j+1}, u_{p-1}) : u_p$ are generated in degree one, for $p = j+1, \dots, m-1$. Since $\deg u_m < \deg u_q$ for $q = j+1, \dots, m-1$, we have $\deg(u_q : u_m) > 1$. Since u_1, \dots, u_m is an admissible order, for any $r \leq j$, there exists a $k \leq j$ such that $\deg(u_k : u_m) = 1$ and $u_k : u_m \mid u_r : u_m$. This shows that $(u_1, \dots, u_j) : u_m$ is generated in degree one. Now let $j+1 \leq p \leq m-1$. It is clear that for any $r \leq p-1$, there exists a $k \leq p-1$ such that $\deg(u_k : u_p) = 1$ and $u_k : u_p \mid u_r : u_p$, since the ideal $(u_1, \dots, u_j, u_{j+1}, \dots, u_p)$ has linear quotients in this order. It remains to show that there is an $h < p$ such that $\deg(u_h : u_p) = 1$ and $u_h : u_p \mid u_m : u_p$. Since $u_1, \dots, u_j, u_{j+1}, \dots, u_m$ is an admissible order and $\deg u_m < \deg u_q$ for $q = j+1, \dots, m-1$, there exists a $k \leq j$ such that $u_k : u_m = x_d$ and $x_d \mid u_p : u_m$ for some $d \in [n]$. Since $u_1, \dots, u_j, u_{j+1}, \dots, u_p$ is an admissible order, there exists an $h < p$ such that $u_h : u_p = x_b$ and $x_b \mid u_k : u_p$ for some $b \in [n]$.

We claim that $x_b \mid u_m : u_p$. In order to prove this we first show that $b \neq d$. Suppose $b = d$. Then we have $x_d = u_k : u_m$ and $x_d = x_b \mid u_k : u_p$. Hence $\deg_{x_d} u_k = \deg_{x_d} u_m + 1$ and $\deg_{x_d} u_k \geq \deg_{x_d} u_p + 1$, where by $\deg_{x_d} u$ we mean the degree of x_d in u . Therefore $\deg_{x_d} u_m \geq \deg_{x_d} u_p$, which is a contradiction, since $x_d \mid u_p : u_m$.

Now since $x_b = u_h : u_p$ and $x_b \mid u_k : u_p$, we have $\deg_{x_b} u_h = \deg_{x_b} u_p + 1$ and $\deg_{x_b} u_k \geq \deg_{x_b} u_p + 1$. On the other hand, since $x_d = u_k : u_m$ and $b \neq d$, we have $\deg_{x_b} u_m \geq \deg_{x_b} u_k \geq \deg_{x_b} u_p + 1 > \deg_{x_b} u_p$. This implies that $x_b \mid u_m : u_p$. \square

If $\sigma = u_1, \dots, u_m$ is any admissible order of $G(I)$, we denote by $\sigma' = u_{i_1}, \dots, u_{i_m}$ the degree increasing admissible order derived from σ as given in Lemma 2.1. The order σ' is called the degree increasing admissible order induced by σ . Attached to an admissible order σ are the sets $q_{u,\sigma}(I)$ as defined in the previous section. We have the following result.

Proposition 2.2. *Let I be a monomial ideal with linear quotients with respect to the admissible order σ of the generators. Then for all $u \in G(I)$ we have*

$$q_{u,\sigma}(I) = q_{u,\sigma'}(I).$$

Proof. Let $\sigma = u_1, \dots, u_m$ and $\sigma' = u_{i_1}, \dots, u_{i_m}$. Suppose $u = u_k$ in σ and $u = u_{i_t}$ in σ' . Let $x_d \in q_{u,\sigma}(I)$, for some $d \in [n]$, then there exists $j < k$ such that $u_j : u_k = x_d$. In

particular, $\deg u_j \leq \deg u_k$. According to the definition of σ' , u_j comes before u_{i_t} and hence $x_d \in q_{u, \sigma'}(I)$.

Conversely, let $x_d \in q_{u, \sigma'}(I)$ for some $d \in [n]$. Then there exists an i_j with $j < t$, such that $u_{i_j} : u_{i_t} = x_d$. We may assume that j is the smallest integer with this property and $u_{i_j} = u_r$ in σ .

Suppose $x_d \notin q_{u, \sigma}(I)$. Then $r > k$ and $\deg u_r < \deg u_k$ according to the definition of σ' . Therefore $u_r = x_d u$ and $u_k = w u$ where u and w are monomials with $\deg w \geq 2$ and $x_d \nmid w$. Since u_1, \dots, u_r is an admissible order and $k < r$, there exists an $s < r$ such that $u_s : u_r = x_b$ and $x_b \mid u_k : u_r = w$ ($b \neq d$). Hence $\deg u_s \leq \deg u_r = \deg u_{i_j}$. Therefore $u_s = u_{i_l}$ with $l < j$.

It follows that $\deg_{x_b} u_s = \deg_{x_b} u_r + 1 \leq \deg_{x_b} u_k$, $\deg_{x_c} u_s \leq \deg_{x_c} u_r \leq \deg_{x_c} u_k$ for any $c \neq d, b$, and $\deg_{x_d} u_s \leq \deg_{x_d} u_r = \deg_{x_d} u_k + 1$. If $\deg_{x_d} u_s < \deg_{x_d} u_k + 1$, then we have $u_s \mid u_k$, a contradiction. Therefore $\deg_{x_d} u_s = \deg_{x_d} u_k + 1$, and hence $x_d = u_s : u_k = u_{i_l} : u_{i_t}$, contradicting the choice of j . \square

Let Δ be a simplicial complex with $\mathcal{F}(\Delta) = \{F_1, \dots, F_m\}$. Then $I_\Delta = \bigcap_{i=1}^m P_{F_i}$ where $P_{F_i} = (x_j : j \notin F_i)$, see [1, Theorem 5.4.1]. It follows from [11, Lemma 1.2] that $I_{\Delta^\vee} = (u_1, \dots, u_m)$, where $u_i = \prod_{j \notin F_i} x_j$. We follow the notation in [2]: if $\delta = F_1, \dots, F_m$ is any order of facets of Δ , then we set $\Delta_k = \langle F_1, \dots, F_k \rangle$ and $R_\delta(F_k) = \{i \in F_k : F_k - \{i\} \in \Delta_{k-1}\}$ for any $k \in [m]$.

We observe the following simple but important fact: Δ is shellable with shelling $\delta = F_1, \dots, F_m$ if and only if I_{Δ^\vee} has linear quotients with the admissible order $\sigma = u_1, \dots, u_m$. Moreover, if the equivalent conditions hold, then $R_\delta(F_k) = q_{u_k, \sigma}(I_{\Delta^\vee})$.

As an immediate consequence of Lemma 2.1, Proposition 2.2 and the observation above we rediscover the following well-known ‘‘Rearrangement Lemma’’ of Björner and Wachs [2, Lemma 2.6].

Corollary 2.3. *Let $\delta = F_1, \dots, F_m$ be a shelling of the simplicial complex Δ . There exists a shelling $\delta' = F_{i_1}, \dots, F_{i_m}$ of Δ induced by δ such that $\dim F_{i_k} \geq \dim F_{i_{k+1}}$ for $k = 1, \dots, m-1$. Furthermore we have $R_\delta(F) = R_{\delta'}(F)$ for any facet F of Δ .*

It is known that the product of two ideals with linear quotients need not to have again linear quotients, even if one of them is generated by linear forms. Such an example was given by Conca and Herzog [3].

Example 2.4. Let $R = k[a, b, c, d]$, $I = (b, c)$ and $J = (a^2b, abc, bcd, cd^2)$. Then J has linear quotients, and I is generated by a subset of the variables. But the product IJ has no linear quotients (not even a linear resolution).

However, we have the following

Lemma 2.5. *Let $I \subset S$ be a monomial ideal. If I has linear quotients, then $\mathfrak{m}I$ has linear quotients, where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of S .*

Proof. We may assume $G(I) = \{u_1, \dots, u_m\}$ and u_1, \dots, u_m is a degree increasing admissible order. We prove the assertion by using induction on m .

The case $m = 1$ is trivial. Let $m > 1$. Consider the multi-set

$$T = \{u_1x_1, \dots, u_1x_n, u_2x_1, \dots, u_2x_n, \dots, u_mx_1, \dots, u_mx_n\}.$$

It is a system of generator of $\mathfrak{m}I$. If $u_i x_j \mid u_r x_s$ for some $i < r$, then we remove $u_r x_s$ from T . In this way, we get the minimal set

$$T' = \{u_i x_j : i = 1, \dots, m, j \in A_i\}$$

of monomial generators of $\mathfrak{m}I$, where $A_1 = [n]$ and $A_i \subset [n]$ for $i = 2, \dots, m$. We shall order $G(\mathfrak{m}I)$ in the following way: $u_k x_l$ comes before $u_t x_s$ if $k < t$ or $k = t$ and $l < s$. Now we show that the above order σ of $G(\mathfrak{m}I)$ is an admissible order. We define the order of the generators of $\mathfrak{m}(u_1 \dots, u_{m-1})$ in the same way as we did for $\mathfrak{m}I$. Then the ordered sequence τ of the generators of $\mathfrak{m}(u_1 \dots, u_{m-1})$ is an initial sequence of σ . Moreover, by induction hypothesis, τ is an admissible order of $G(\mathfrak{m}(u_1 \dots, u_{m-1}))$.

For a given $j \in A_m$ let J be the ideal generated by all monomials in T' which come before $u_m x_j$ with respect to σ . It remains to be shown that $J : u_m x_j$ is generated by monomials of degree 1.

Let $u_k x_l \in G(J)$. If $k = m$, then $u_k x_l : u_m x_j = x_l$. If $k < m$, then we shall find an element $u_r x_s \in G(J)$ and $t \in [n]$ such that $u_r x_s : u_m x_j = x_t$ and $x_t \mid u_k x_l : u_m x_j$. Indeed since u_1, \dots, u_m is an admissible order of $G(I)$, there exists $q < m$ such that $u_q : u_m = x_t$ and $x_t \mid u_k : u_m$. This implies that $u_q x_j : u_m x_j = u_q : u_m = x_1$. Since $u_q x_j \in \mathfrak{m}I$, there exists, by the definition of σ , a monomial $u_r x_s \in G(J)$ such that $u_r x_s \mid u_q x_j$.

We claim that $u_r x_s : u_m x_j = x_t$ and $x_t \mid u_k x_l : u_m x_j$. Notice that $u_r x_s : u_m x_j \mid u_q x_j : u_m x_j = x_t$. If $u_r x_s : u_m x_j \neq x_t$, then $u_r x_s : u_m x_j = 1$, that is, $u_r x_s \mid u_m x_j$ which contradicts the fact that $j \in A_m$. This shows that $u_r x_s : u_m x_j = x_t$.

Since $x_t \mid u_k : u_m$, it is enough to show that $x_t \neq x_j$ in order to prove that $x_t \mid u_k x_l : u_m x_j$. Assume that $x_t = x_j$. Since $u_q : u_m = x_t$, we have $u_q = x_t u$ for some monomial u such that $u \mid u_m$. Since $\deg u_q \leq \deg u_m$, it follows that $u_m = uw$ for some monomial w with $\deg w \geq 1$ and $x_t \nmid w$. Hence there exists some variable x_d with $d \neq t$ such that $x_d \mid w$. But then $x_d u_q = x_d u x_t \mid w u x_t = u_m x_j$, contradicting again the fact that $j \in A_m$. \square

Remark 2.6. The converse of the above lemma is not true. For example, let $I = (ab, cd) \subset K[a, b, c, d]$. Then $\mathfrak{m}I = (a^2 b, ab^2, abc, abd, acd, bcd, c^2 d, cd^2)$ has linear quotients in the given order, but I has no linear quotients.

Now we present the main theorem of this section.

Theorem 2.7. *Let $I \subset S$ be a monomial ideal. If I has linear quotients, then I has componentwise linear quotients.*

Proof. By Lemma 2.5 and Lemma 2.1, we may assume that I is generated by monomials of two different degrees a and $a + 1$. We denote by $I_{\langle a \rangle}$ the ideal generated by the a -th graded component of the ideal I . Let $G(I) = \{u_1, \dots, u_s, v_1, \dots, v_t\}$, where $\deg u_i = a$ for $i = 1, \dots, s$ and $\deg v_j = a + 1$ for $j = 1, \dots, t$. By Lemma 2.1, we may assume that $u_1, \dots, u_s, v_1, \dots, v_t$ is an admissible order, hence I_a has linear quotients. Now we show that $I_{\langle a+1 \rangle}$ has also linear quotients.

We have $I_{\langle a+1 \rangle} = \mathfrak{m}(u_1, \dots, u_s) + (v_1, \dots, v_t)$. Let $G(I_{\langle a+1 \rangle}) = \{w_1, \dots, w_l, v_1, \dots, v_t\}$, where w_1, \dots, w_l is ordered as in Lemma 2.5. In particular, w_1, \dots, w_l is an admissible order. We only need to show that $(w_1, \dots, w_l, v_1, \dots, v_{p-1}) : v_p$ is generated by a subset of the variables, for $1 \leq p \leq t$.

First we consider $v_j : v_p$ where $j < p$. Since $u_1, \dots, u_s, v_1, \dots, v_t$ is an admissible order of $G(I)$, there exists some $u \in \{u_1, \dots, u_s, v_1, \dots, v_t\}$ and $d \in [n]$ such that $u : v_p = x_d$ and $x_d \mid v_j : v_p$. If $u \in \{v_1, \dots, v_t\}$ we are done. So we may assume $u \in \{u_1, \dots, u_s\}$. Therefore, $\deg u = \deg v_p - 1$. Since $u : v_p = x_d$, $\deg_{x_d} u = \deg_{x_d} v_p + 1$ and $\deg_{x_b} u \leq \deg_{x_b} v_p$ for any $b \neq d$. Since $\deg u < \deg v_p$, there exists a variable x_c with $c \neq d$ such that $\deg_{x_c} u \leq \deg_{x_c} v_p - 1$. Since $x_c u \in \mathfrak{m}I_{\langle a \rangle}$, one has $x_c u = w_k$ for some $k \leq l$. All this implies that $\deg_{x_d} w_k = \deg_{x_d} u = \deg_{x_d} v_p + 1$ and $\deg_{x_b} w_k \leq \deg_{x_b} v_p$ for any $b \neq d$. Therefore $w_k : v_p = x_d$ and $x_d \mid v_j : v_p$.

It remains to consider $w_j : v_p$. In this case $w_j = x_b u_i$ for some $i \in [s]$ and some $b \in [n]$. Since $u_1, \dots, u_s, v_1, \dots, v_t$ is an admissible order, there exists some $u \in \{u_1, \dots, u_s, v_1, \dots, v_t\}$ and $d \in [n]$ such that $u : v_p = x_d$ and $x_d \mid u_i : v_p$. Therefore $x_d \mid w_j : v_p$, since $u_i : v_p \mid w_j : v_p$. If $u \in \{v_1, \dots, v_t\}$, then we are done. So we may assume $u \in \{u_1, \dots, u_s\}$. Then, as before, there exists a variable x_c with $c \neq d$ such that $x_c u \in \mathfrak{m}I_{\langle a \rangle}$, $\deg_{x_d} x_c u = \deg_{x_d} u = \deg_{x_d} v_p + 1$ and $\deg_{x_b} x_c u \leq \deg_{x_b} v_p$ for any $b \neq d$. This implies that $x_c u : v_p = x_d$ and $x_d \mid w_j : v_p$. \square

Corollary 2.8. *If $I \subset S$ is a monomial ideal with linear quotients, then I is componentwise linear.*

We do not know if the converse of Theorem 2.7 is true in general. However we could prove the following:

Proposition 2.9. *Let I be a monomial ideal with componentwise linear quotients. Suppose for each component $I_{\langle a \rangle}$ there exists an admissible order σ_a of $G(I_{\langle a \rangle})$ with the property that the elements of $G(\mathfrak{m}I_{\langle a-1 \rangle})$ form the initial part of σ_a . Then I has linear quotients.*

Proof. We chose the order $\sigma = u_1, \dots, u_s$ of $G(I)$ such that that $i < j$ if $\deg u_i < \deg u_j$ or $\deg u_i = \deg u_j = a$ and u_i comes before u_j in σ_a .

We show that $(u_1, \dots, u_{p-1}) : u_p$ is generated by linear forms. If $\deg u_1 = \deg u_p$, then there is nothing to prove.

Now assume that $\deg u_1 < \deg u_p = b$. Let $l < p$ be the largest number such that $\deg u_l < b$. Then, by our assumption, there exists an admissible order $w_1, \dots, w_t, u_{l+1}, \dots, u_p$ where $w_1, \dots, w_t \in G(\mathfrak{m}I_{\langle b-1 \rangle})$.

Let $j < p$ and suppose that $\deg(u_j : u_p) \geq 2$. Let m be a monomial such that $\deg(mu_j) = \deg u_p$ and $mu_j : u_p = u_j : u_p$. Since $mu_j \in \{w_1, \dots, w_t, u_{l+1}, \dots, u_{p-1}\}$ there exists $w \in \{w_1, \dots, w_t, u_{l+1}, \dots, u_{p-1}\}$ and some $d \in [n]$ such that $w : u_p = x_d$ and $x_d \mid u_j : u_p$ because $mu_j : u_p = u_j : u_p$.

If $w \in \{u_{l+1}, \dots, u_{p-1}\}$, then we are done. On the other hand, if $w \in \{w_1, \dots, w_t\}$, then $w = m' u_i$ for some $i \leq l$ and some monomial m' . Since $w : u_p = x_d$, one has $\deg_{x_b} w \leq \deg_{x_b} u_p$ for all $b \neq d$. Hence x_d does not divide m' , otherwise $u_i \mid u_p$ which contradicts the fact that $u_i, u_p \in G(I)$. Therefore $x_d = u_i : u_p$ and $x_d \mid u_j : u_p$. \square

Let $I \subset S$ be a monomial ideal. We denote by I^* the monomial ideal generated by the squarefree monomials in I and call it the squarefree part of I . Indeed $I^* = (u : u \in G(I) \text{ and } u \text{ is squarefree})$. We follow [8] and denote by $I_{[a]}$ the squarefree part of $I_{\langle a \rangle}$. In [8, Proposition 1.5], the authors proved that if I is squarefree, then $I_{\langle a \rangle}$ has a linear

resolution if and only if $I_{[a]}$ has a linear resolution. Indeed for the “only if” part one does not need the assumption that I is squarefree. We have the following slightly different result.

Proposition 2.10. *Let I be a monomial ideal in S . If I has linear quotients, then I^* has linear quotients.*

Proof. Let u_1, \dots, u_m be an admissible order of $G(I)$. Assume $I^* = (u_{i_1}, \dots, u_{i_t})$, where $1 \leq i_1 < i_2 < \dots < i_t \leq m$. We shall show u_{i_1}, \dots, u_{i_t} is an admissible order of $G(I^*)$ by using induction on m .

The case $m = 1$ is trivial. Now assume $m > 1$. It is clear that $(u_{i_1}, \dots, u_{i_{t-1}})$ is the squarefree part of the monomial ideal $(u_1, \dots, u_{i_{t-1}})$, where $u_1, \dots, u_{i_{t-1}}$ is an admissible order. By induction hypothesis $u_{i_1}, \dots, u_{i_{t-1}}$ is an admissible order of $G((u_{i_1}, \dots, u_{i_{t-1}}))$. Consider $u_{i_j} : u_{i_t}$ with $j < t$. Since u_1, \dots, u_m is an admissible order of $G(I)$, there exists $k < i_t$ and some $d \in [n]$ such that $u_k : u_{i_t} = x_d$ and $x_d \mid u_{i_j} : u_{i_t}$. Since u_{i_j} and u_{i_t} are squarefree, we have $x_d \nmid u_{i_t}$. On the other hand, since $u_k : u_{i_t} = x_d$, one has $\deg_{x_d} u_k = 1$ and $\deg_{x_b} u_k \leq \deg_{x_b} u_{i_t} \leq 1$ for any $b \neq d$. Hence $u_k \in G(I^*)$. \square

Combining Proposition 2.10 with Theorem 2.7, we obtain:

Corollary 2.11. *Let $I \subset S$ be a monomial ideal with linear quotients. Then $I_{[a]}$ has linear quotients for all a .*

Remark 2.12. All results concerning linear quotients proved in this section are correspondingly valid for monomial ideals in the exterior algebra.

Let Δ be a d -dimensional simplicial complex. We define the 1-*facet skeleton* of Δ to be the simplicial complex

$$\Delta^{[1]} = \langle G : G \subset F \in \mathcal{F}(\Delta) \text{ and } |G| = |F| - 1 \rangle.$$

Recursively, the i -*facet skeleton* is defined to be the 1-facet skeleton of $\Delta^{[i-1]}$, for $i = 1, \dots, d$. For example if $\Delta = \langle \{1, 2, 3\}, \{2, 3, 4\}, \{4, 5\} \rangle$, then

$$\Delta^{[1]} = \langle \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{5\} \rangle \text{ and } \Delta^{[2]} = \langle \{1\}, \{2\}, \{3\}, \{4\} \rangle.$$

If Δ is pure of dimension d , then the i -facet skeleton of Δ is just the $(d - i)$ -skeleton of Δ . Now let Γ be a shellable simplicial complex with facets F_1, \dots, F_m . It is known that any skeleton of Γ is shellable, see [2, Theorem 2.9]. Since $I_\Gamma = \bigcap_{i=1}^m P_{F_i}$ where $P_{F_i} = (x_j : j \notin F_i)$, we have $(I_\Gamma)^\vee = (u_1, \dots, u_m)$, where $u_i = \prod_{j \notin F_i} x_j$. By Theorem 1.4 $(I_\Gamma)^\vee$ has linear quotients. Hence $\mathfrak{m}(I_\Gamma)^\vee$ and the squarefree part of $\mathfrak{m}(I_\Gamma)^\vee$ have linear quotients by Lemma 2.5 and Proposition 2.10. It is not hard to see that the squarefree part of $\mathfrak{m}(I_\Gamma)^\vee$ is the Alexander dual of $I_{\Gamma[1]}$. Hence our discussions yield the following:

Corollary 2.13. *If Γ is a shellable simplicial complex of dimension d , then $\Gamma^{[i]}$ is shellable, for $i \leq d$. In particular, if Γ is pure, then any skeleton of Γ is again shellable.*

3. A CLASS OF PRETTY CLEAN MONOMIAL IDEALS

In this section we study a class of monomial ideals which are pretty clean. This class is a generalization of the class of facet ideals of forests.

Let $I \subset S$ be a squarefree monomial ideal. There is a unique simplicial complex Δ such that $I = I(\Delta)$. Now we generalize the concept of the facet ideal of a forest as follows: Let I be a monomial ideal (not necessarily squarefree) with $G(I) = \{u_1, \dots, u_m\}$. A variable x_i is called a free variable of I if there exists a $t \in [m]$ such that $x_i \mid u_t$ and $x_i \nmid u_j$ for any $j \neq t$. A monomial u_t is called a *leaf* of $G(I)$ if u_t is the only element in $G(I)$ or there exists a $j \in [m]$, $j \neq t$ such that $\gcd(u_t, u_i) \mid \gcd(u_t, u_j)$ for all $i \neq t$. In this case u_j is called a *branch* of u_t . We say that I is a *monomial ideal of forest type* if any subset of $G(I)$ has a leaf. It is clear that any monomial ideal of forest type has a free variable.

Let $(X_1, X_2) = (\{x_{i_1}, \dots, x_{i_r}\}, \{x_{j_1}, \dots, x_{j_s}\})$, where X_1, X_2 are subsets of $X = \{x_1, \dots, x_n\}$ and $X_1 \cap X_2 = \emptyset$. Let I be a monomial ideal in $S = K[x_1, \dots, x_n]$. As in [23] we define the *minor* of I with respect to (X_1, X_2) to be the ideal $I_{(X_1, X_2)} \subset K[X \setminus (X_1 \cup X_2)]$ obtained from I by setting $x_{i_k} = 0$ for $k = 1, \dots, r$ and $x_{j_l} = 1$ for $l = 1, \dots, s$. In particular, $I_{(\emptyset, \emptyset)} = I$. One says that the ideal I has the *free variable property* if all minors of I have free variables. The following lemma is a generalization of [6, Lemma 4.5] to any monomial ideal of forest type.

Lemma 3.1. *Let I be a monomial ideal of forest type and $X' = \{x_{j_1}, \dots, x_{j_s}\}$ a subset of X . Then $I_{(\emptyset, X')}$ is again a monomial ideal of forest type.*

Proof. We only need to prove that $I_{(\emptyset, \{x_{j_1}\})}$ is a monomial ideal of forest type. Hence we may assume that $X' = \{x_i\}$. Let $G(I) = \{u_1, \dots, u_m\}$. We write $u_j = \bar{u}_j x_i^{a_j}$, where $a_j \geq 0$ and $x_i \nmid \bar{u}_j$ for $j = 1, \dots, m$. Let A be any subset of $G(I_{(\emptyset, X')})$. Consider the subset $A' = \{u_j : \bar{u}_j \in A\}$ of $G(I)$. Since I is a monomial ideal of forest type, A' has a leaf u_p . This means that there exists a $u_k \in A'$ such that $\gcd(u_p, u_q) \mid \gcd(u_p, u_k)$ for all $u_q \in A'$ with $q \neq p$.

Let $\gcd(u_p, u_q) = v_q x_i^{a_q}$ and $\gcd(u_p, u_k) = v_k x_i^{a_k}$, where v_q, v_k are monomials and $x_i \nmid v_q, x_i \nmid v_k$. Then $\gcd(\bar{u}_p, \bar{u}_q) = v_q$ which divides $\gcd(\bar{u}_p, \bar{u}_k) = v_k$ for all $\bar{u}_q \in A$ with $q \neq p$. Hence \bar{u}_p is a leaf of A . \square

Now we recall the following fact from [16], which is needed for the proof of the next proposition.

Lemma 3.2. *Let $K \subset S$ be a monomial ideal and u a monomial in S which is regular over S/K . If S/K is pretty clean, then $S/(K, u)$ is pretty clean.*

The following proposition is crucial for proving one of the main results of this section.

Proposition 3.3. *Let $I \subset S$ be a monomial ideal with $G(I) = \{u_1, \dots, u_{m-1}, \bar{u}_m x_j^t\}$ where x_j is a free variable of I and $x_j \nmid \bar{u}_m$. If $I_{(\emptyset, \{x_j\})}$ and $I_{(\{x_j\}, \emptyset)}$ are pretty clean, then I is pretty clean.*

Proof. We denote $I_{(\emptyset, \{x_j\})} = (u_1, \dots, u_{m-1}, \bar{u}_m)$ and $I_{(\{x_j\}, \emptyset)} = (u_1, \dots, u_{m-1})$ by J and K respectively. It is easy to see that $J/I = (I, \bar{u}_m)/I \cong S/(I : \bar{u}_m) = S/(K, x_j^t)$. Since S/K is pretty clean, by Lemma 3.2 J/I is also pretty clean. Let $\mathcal{F}_1 : I = I_0 \subset I_1 \subset \dots \subset I_r = J$ be a

pretty clean filtration of J/I with $I_i/I_{i-1} \cong S/P_i$. Then by [13, Corollary 3.4] $\text{Supp}(\mathcal{F}_1) = \text{Ass}(J/I) = \text{Ass}(S/(K, x_j^t))$. Hence $x_j \in P_i$ for $i = 1, \dots, r$.

By our assumption S/J is pretty clean. Let $\mathcal{F}_2: J = I_r \subset I_{r+1} \subset \dots \subset I_{r+s} = S$ be a pretty clean filtration of S/J with $I_{r+i}/I_{r+i-1} \cong S/P_{r+i}$. Then $P_{r+i} \in \text{Ass}(S/J)$. Hence $x_j \notin P_{r+i}$ for $i = 1, \dots, s$.

Combining the prime filtrations \mathcal{F}_1 and \mathcal{F}_2 we get the prime filtration

$$\mathcal{F}: I = I_0 \subset \dots \subset I_r = J \subset I_{r+1} \subset \dots \subset I_{r+s} = S$$

of S/I . Since $x_j \in P_i$ for $i = 1, \dots, r$ and $x_j \notin P_{r+i}$ for $i = 1, \dots, s$, one has $P_i \not\subseteq P_{r+t}$ for any $i \in [r]$ and any $t \in [s]$. Therefore \mathcal{F} is a pretty clean filtration of S/I since \mathcal{F}_1 and \mathcal{F}_2 are pretty clean filtrations. \square

Combining Proposition 3.3 with Lemma 3.1, we get the following theorem.

Theorem 3.4. *If $I \subset S$ is a monomial ideal of forest type, then S/I is pretty clean.*

Proof. We use induction on n the number of variables to prove the assertion. Let $G(I) = \{u_1, \dots, u_m\}$ and let x_i be a free vertex of I . We may assume that $u_m = \bar{u}_m x_i^a$ with $a > 0$. By Lemma 3.1, the ideal $J = (u_1, \dots, u_{m-1}, \bar{u}_m)$ is a monomial ideal of forest type. It is clear that $K = (u_1, \dots, u_{m-1})$ is also a monomial ideal of forest type. By induction hypothesis S/J and S/K are pretty clean. Therefore by Proposition 3.3, S/I is pretty clean. \square

Let $I \subset S$ be a monomial ideal. According to Stanley [21, Section II, 3.9] and Schenzel [17], a finite filtration $\mathcal{F}: I = I_0 \subset I_1 \subset \dots \subset I_r = S$ of S/I is called a *Cohen–Macaulay filtration* if each quotient I_j/I_{j-1} is Cohen–Macaulay, and

$$\dim(I_1/I_0) < \dim(I_2/I_1) < \dots < \dim(I_r/I_{r-1}).$$

The module S/I is called *sequentially Cohen–Macaulay* if it has a Cohen–Macaulay filtration.

It follows from [13, Corollary 4.3] that if S/I is pretty clean, then S/I is sequentially Cohen–Macaulay. Therefore we have the following corollary, which generalizes the main result of Faridi [7].

Corollary 3.5. *If $I \subset S$ is a monomial ideal of forest type, then S/I is sequentially Cohen–Macaulay.*

Let I be a monomial ideal, any decomposition of S/I as direct sum of K -vector spaces of the form $uK[Z]$, where u is a monomial in S and $Z \subset \{x_1, \dots, x_n\}$, is called a *Stanley decomposition* of S/I . Stanley conjectured in [20] that there always exists a Stanley decomposition

$$S/I = \bigoplus_{i=1}^r u_i K[Z_i]$$

such that $|Z_i| \geq \text{depth}(S/I)$ for all $i \in [r]$. Recently Stanley's conjecture was studied in several articles, see for example [13], [14] and [19].

In [13, Theorem 6.5], the authors proved the following

Theorem 3.6. *Let $I \subset S$ be a monomial ideal. If I is pretty clean, then Stanley's conjecture holds for S/I .*

As an immediate consequence of Theorem 3.4 and Theorem 3.6 we have the following:

Corollary 3.7. *If I is a monomial ideal of forest type, then Stanley's conjecture holds for S/I .*

Let \mathcal{J} be the class of monomial ideals with the following properties:

- (a) any irreducible monomial ideal is in \mathcal{J} ;
- (b) each $I \in \mathcal{J}$ has a free variable;
- (c) if x_i is a free variable of I , then $I \in \mathcal{J}$ if and only if the minors $I_{(\emptyset, \{x_i\})}$ and $I_{(\{x_i\}, \emptyset)}$ are in \mathcal{J} .

It is obvious that if a monomial ideal I has the free variable property, then $I \in \mathcal{J}$. Moreover we have the following:

Theorem 3.8. *Let $I \subset S$ be a monomial ideal. The following statements are equivalent.*

- (i) *I is a monomial ideal of forest type;*
- (ii) *I has the free variable property;*
- (iii) *$I \in \mathcal{J}$.*

Proof. (i) \Rightarrow (ii): Let X_1 and X_2 be any subsets of X with $X_1 \cap X_2 = \emptyset$. Since any monomial ideal J with $G(J) \subset G(I)$ is again a monomial ideal of forest type, this together with Lemma 3.1 imply that $I_{(X_1, X_2)}$ is again a monomial ideal of forest type. Hence it has a free variable.

(ii) \Rightarrow (iii) is obvious.

(iii) \Rightarrow (i): We show that I is a monomial ideal of forest type by using induction on the number of variables n which appear in I . The case $n = 1$ is clear. Let $n > 1$. Since $I \in \mathcal{J}$, we may assume that $G(I) = \{u_1, \dots, u_m\}$, where $u_m = \bar{u}_m x_t^a$ and x_t is a free variable of I . Since the ideals $J = (u_1, \dots, u_{m-1}, \bar{u}_m)$ and $K = (u_1, \dots, u_{m-1})$ are in \mathcal{J} with less variables, by induction hypothesis J and K are monomial ideals of forest type. Let A be any subset of $G(I)$. If $u_m \notin A$, then $A \subset G(K)$. Hence it has a leaf. If $u_m \in A$ and $\bar{u}_m \mid u_j$ for some $u_j \in A$ and $j \neq m$, then $\gcd(u_m, u_j) = \bar{u}_m$ and $\gcd(u_m, u_i) \mid \bar{u}_m$ for any $i \neq m$. This means that u_m is a leaf of A . Now we may assume that $u_m \in A$ and $\bar{u}_m \nmid u_j$ for any $u_j \in A$ and $j \neq m$. Then $A' = (A \setminus \{u_m\}) \cup \{\bar{u}_m\}$ is a subset of $G(J)$ and hence it has a leaf. Let u_p be a leaf of A' . Since x_t is a free variable, we have $\gcd(u_m, u_i) = \gcd(\bar{u}_m, u_i)$ for any $i \neq m$. If $u_p = \bar{u}_m$, then u_m is a leaf of A . If $u_p \neq \bar{u}_m$, then u_p itself is a leaf of A . \square

A clutter \mathcal{C} with vertex set $[n]$ is a family of subsets of $[n]$, called *edges*, with the property that non of them is contained in another. The edge ideal of a clutter \mathcal{C} is defined to be the ideal $I(\mathcal{C}) = (x_C : C \text{ is an edge of } \mathcal{C})$, where $x_C = \prod_{i \in C} x_i$. A clutter is a special kind of hypergraph. One may also view a clutter \mathcal{C} as the set of facets of some simplicial complex Δ . In this case, $I(\mathcal{C}) = I(\Delta)$.

In [23], the authors say a clutter \mathcal{C} has the free vertex property if the edge ideal $I(\mathcal{C})$ has the free variable property. By Theorem 3.8 one sees that \mathcal{C} has the free vertex property if and only if $I(\mathcal{C})$ is a monomial ideal of forest type. If we consider \mathcal{C} to be the set of facets of some simplicial complex Δ , then \mathcal{C} has the free vertex property if and only if Δ is a forest. In the following we denote by $\Delta_{\mathcal{C}}$ the simplicial complex whose Stanley–Reisner ideal is $I(\mathcal{C})$.

As a corollary of Theorem 3.8 and Theorem 3.4, we obtain the following:

Corollary 3.9. ([23, Theorem 5.3]) *If the clutter \mathcal{C} has the free vertex property, then $S/I(\mathcal{C})$ is clean, i.e. $\Delta_{\mathcal{C}}$ is shellable.*

Let \mathcal{C} be a clutter and Δ the simplicial complex such that $I(\mathcal{C}) = I(\Delta)$. We say that the clutter \mathcal{C} is a forest if Δ is a simplicial forest. Up to the order of the vertices and the order of the edges, a clutter is determined by its incidence matrix and vice versa. The incidence matrix $M_{\mathcal{C}}$ is defined as follows: let $1, \dots, n$ be the vertices and C_1, \dots, C_m be the edges of the clutter \mathcal{C} . Then $M_{\mathcal{C}} = (e_{ij})$ is an $n \times m$ matrix with $e_{ij} = 1$ if $i \in C_j$ and $e_{ij} = 0$ if $i \notin C_j$. A clutter is called *totally balanced* if its incidence matrix has no square submatrix of order at least 3 with exactly two 1's in each row and column. It is known that a totally balanced clutter has the free vertex property, see [18, Corollary 83.3a]. On the other hand, in [10, Theorem 3.2], it is shown that \mathcal{C} is a forest if and only if \mathcal{C} is totally balanced. These together with Theorem 3.8 imply the following:

Corollary 3.10. *Let \mathcal{C} be a clutter. The following statements are equivalent:*

- (i) \mathcal{C} is a forest;
- (ii) \mathcal{C} is totally balanced;
- (iii) \mathcal{C} has the free vertex property.

To end this section, we would like to mention that if I is the facet ideal of some forest Δ , then I is a monomial ideal of forest type. Hence S/I is clean. By Corollary 1.5, I^{\vee} has linear quotients.

4. SOME EXAMPLES AND QUESTIONS

In Sections 3, we show that the facet ideal I of any forest is clean and hence Stanley's conjecture holds for S/I . There is a more general class of simplicial complexes, the class of quasi-forests. It is natural to ask whether the facet ideal of any quasi-forest is again clean?

According to [25], a connected simplicial complex Δ is called a *quasi-tree*, if there exists an order F_1, \dots, F_m of the facets, such that F_i is a leaf of $\langle F_1, \dots, F_i \rangle$ for each $i = 1, \dots, m$. Such an order is called a *leaf order*. A simplicial complex Δ with the property that every connected component is a quasi-tree is called a *quasi-forest*. It is clear that any forest is a quasi-forest.

Unfortunately the facet ideal of a quasi-forest need not to be clean. For example the facet ideal of the quasi-tree $\Gamma = \langle \{1, 2, 3, 4\}, \{1, 4, 5\}, \{1, 2, 8\}, \{2, 3, 7\}, \{3, 4, 6\} \rangle$, as in Figure 1, is not clean. Indeed

$$I(\Gamma)^{\vee} = (x_1x_3, x_2x_4, x_4x_7x_8, x_1x_6x_7, x_1x_4x_7, x_2x_3x_5, x_1x_2x_6, x_2x_5x_6, x_3x_4x_8, x_3x_5x_8)$$

has no linear quotients, even no componentwise linear quotients.

One might expect that the facet ideal of any quasi-forest which is not a forest is not clean. The following example shows that this is not the case. The facet ideal of the quasi-tree $\Gamma' = \langle \{1, 2, 3\}, \{2, 4, 5\}, \{2, 3, 5\}, \{3, 5, 6\} \rangle$, as in Figure 2, is clean. Since $I(\Gamma')^{\vee} = (x_3x_5, x_2x_5, x_1x_5, x_2x_6, x_2x_3, x_3x_4)$ has linear quotients in the given order.

It would be interesting to classify all quasi-forests such that their facet ideals are clean.

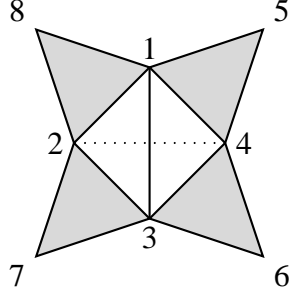


FIGURE 1.

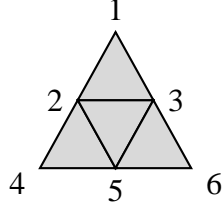


FIGURE 2.

Even though $I(\Gamma)$ (Γ is the quasi-tree as given in Figure 1) is not clean we will show that Stanley's conjecture holds for $S/I(\Gamma)$. First we recall some notation and results from [14].

Let Δ be a simplicial complex on the vertex set $[n]$. A subset $\mathcal{I} \subset \Delta$ is called an *interval* in Δ , if there exists faces $F, G \in \Delta$ such that $\mathcal{I} = \{H \in \Delta: F \subseteq H \subseteq G\}$. We denote this interval by $[F, G]$. A *partition* \mathcal{P} of Δ is a presentation of Δ as a disjoint union of intervals in Δ . Stanley calls a simplicial complex Δ *partitionable* if there exists a partition $\Delta = \bigcup_{i=1}^r [F_i, G_i]$ with $\mathcal{F}(\Delta) = \{G_1, \dots, G_r\}$ and conjectured [21, Conjecture 2.7] (see also [22, Problem 6]) that each Cohen-Macaulay simplicial complex is partitionable. It follows from [14, Corollary 3.5] that the conjecture on Stanley decompositions implies the conjecture on partitionable simplicial complexes.

For $F \subseteq [n]$ we set $x_F = \prod_{i \in F} x_i$ and $Z_F = \{x_i: i \in F\}$. It follows from [14, Proposition 3.2] that if $\bigcup_{i=1}^r [F_i, G_i]$ is a partition of Δ , then $\bigoplus_{i=1}^r x_{F_i} K[Z_{G_i}]$ is a Stanley decomposition of S/I_Δ .

Now let Δ be the simplicial complex with the property that $I_\Delta = I(\Gamma)$. Then Stanley's conjecture holds for $S/I(\Gamma)$, if there is a partition $\bigcup_{i=1}^r [F_i, G_i]$ of Δ such that $|G_i| \geq \text{depth } S/I_\Delta$ for all i .

The facets of Δ are

$$\begin{aligned} &\{1, 3, 5, 6, 7, 8\}, \{2, 4, 5, 6, 7, 8\}, \{1, 2, 3, 5, 6\}, \{2, 3, 4, 5, 8\}, \{2, 3, 5, 6, 8\}, \\ &\{1, 4, 6, 7, 8\}, \{3, 4, 5, 7, 8\}, \{1, 3, 4, 7, 8\}, \{1, 2, 5, 6, 7\} \text{ and } \{1, 2, 4, 6, 7\}. \end{aligned}$$

Consider the partition

$$\begin{aligned} \mathcal{P} = & [\emptyset, 135678] \cup [2, 12356] \cup [4, 245678] \cup [14, 14678] \cup [27, 12567] \cup [34, 34578] \\ & \cup [28, 23568] \cup [124, 12467] \cup [134, 13478] \cup [234, 23458] \cup [278, 25678] \end{aligned}$$

of Δ . Here 135678 stands for the set $\{1, 3, 5, 6, 7, 8\}$, and a similar notation is used for the other sets.

\mathcal{P} has the property that the cardinality of upper face of each interval is greater than or equal to

$$\min\{|F| : F \text{ is a facet of } \Delta\} \geq \text{depth}(S/I_\Delta) = \text{depth}(S/I(\Gamma)).$$

This shows that Stanley's conjecture holds for $S/I(\Gamma)$.

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